Math 2002 Final: Solutions

1. (a) Fubini's theorem states that if f is continuous on a rectangle $R = [a,b] \times [c,d]$, then

$$\int_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

   (b) The fundamental theorem of line integrals says that if f is a continuous function, then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

   where C has parametrization $r(t)$, $a \leq t \leq b$.

   (c) Stokes' theorem says that if C is the boundary curve of a smooth surface S, and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives, then

$$\int_C \mathbf{F} \cdot dr = \int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

2. (a)

$$\begin{aligned}
& \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\ dz\,dy\,dx \\
= & \int_0^1 \int_0^{1-x} x(1-x) - xy\,dy\,dx \\
= & \int_0^1 \left(x(1-x)y - \frac{xy^2}{2} \right)_0^{1-x} dx \\
= & \int_0^1 x(1-x)^2 - \frac{x(1-x)^2}{2}\,dx \\
= & \int_0^1 \frac{x(1-x)^2}{2}\,dx \\
= & \frac{1}{2}\int_0^1 x - 2x^2 + x^3\,dx \\
= & \frac{1}{2}\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right)_0^1 \\
= & \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\
= & \frac{1}{24}
\end{aligned}$$

(b) C has parametrization $r(t) = (-1-2t, 2+3t)$, $0 \leq t \leq 1$. Thus

$$\begin{aligned}
& \int_C y^2 + e^x \, dx \\
= \quad & \int_0^1 \left((2+3t)^2 + e^{-1-2t}\right)(-2)\, dt \\
= \quad & -2\int_0^1 4 + 12t + 9t^2 + e^{-1-2t}\, dt \\
= \quad & -2\left(4t + 6t^2 + 3t^3 + \frac{-1}{2}e^{-1-2t}\right)_0^1 \\
= \quad & -2\left(4 + 6 + 3 + \frac{-e^{-3}}{2} + \frac{e^{-1}}{2}\right) \\
= \quad & -26 + e^{-3} - e^{-1}
\end{aligned}$$

3. The complementary equation $y'' + 6y' + 9y = 0$ has auxiliary equation $x^2+6x+9 = 0$, which factors as $(x+3)^2 = 0$. Thus the complementary equation has solution

$$y_c = c_1 e^{-3x} + c_2 x e^{-3x}$$

To find a particular solution, we use the method of undetermined coefficients, with $y_p = Ax + B$. This has derivative $y_p' = A$. Putting this into the given DE gives

$$6A + 9Ax + 9B = 3x + 1$$

Thus $9A = 3$ and $6A + 9B = 1$. Solving for A and B gives $A = \frac{1}{3}$, $B = -\frac{1}{9}$. Thus the DE has general solution

$$y = y_c + y_p = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{1}{3}x - \frac{1}{9}$$

Using $y(0) = 0$ gives $c_1 = \frac{1}{9}$. To use $y'(0)$, we need to find the derivative of y. This is

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x} + \frac{1}{3}$$

Thus $y'(0) = 1$ gives

$$-3c_1 + c_2 + \frac{1}{3} = 1$$

Since $c_1 = \frac{1}{9}$, this gives $c_2 = 1$. Thus the solution is

$$y = \frac{1}{9}e^{-3x} + xe^{-3x} + \frac{1}{3}x - \frac{1}{9}$$

4. (a) One can calculate

$$\text{curl}(\mathbf{F}) = (2z\cos y - 2z\cos y)\mathbf{i} - (0-0)\mathbf{j} + (-\sin y + \sin y)\mathbf{k} = \mathbf{0}$$

so $\mathbf{F}$ is conservative.

(b) We know

$$\frac{\partial f}{\partial x} = \cos y, \frac{\partial f}{\partial y} = z^2\cos y - x\sin y, \frac{\partial f}{\partial z} = 2z\sin y + 3$$

Integrating the first expression with respect to x gives $f = x\cos y + g(y,z)$. Differentianting with respect to y gives

$$\frac{\partial f}{\partial y} = -x\sin y + \frac{\partial g}{\partial y}$$

Comparing with above gives $\frac{\partial g}{\partial y} = z^2\cos y$, so $g = z^2\sin y + h(z)$. Now differentiating f with respect to z gives

$$\frac{\partial f}{\partial z} = 2z\sin y + h'(z)$$

So $h'(z) = 3$, and thus $h(z) = 3z$. So

$$f = x\cos y + z^2\sin y + 3z$$

(c) Since $\mathbf{F}$ is conservative, by the fundamental theorem of line integrals,

$$\int_C \mathbf{F}\cdot dr = f(r(b)) - f(r(a)) = f(1,\pi,2) - f(0,0,1) = 2$$

5. Since the curve is closed, we can use Green's theorem. The region D that the curve bounds is the first eighth of the circle $x^2 + y^2 = 1$, and C is oriented negatively. Thus

$$\int_C \mathbf{F}\cdot dr = -\int_D \frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y}\,dA = -\int_D 4\,dA = -4\int_D 1\,dA = -4\left(\frac{\pi}{8}\right) = \frac{-\pi}{2}$$

6. We can rewrite the equations for u and v as

$$x = v + 4u, y = 2u$$

This has Jacobian -2. The region D is bounded by the curves

$$y = \frac{1}{2}x, \ x = 4, \ y = 0$$

Under the change of variables, this changes to

$$u = 0, v = 0, 4u + v = 4$$

This D region is a triangle. Thus the integral becomes

$$\int_D \sqrt{v} + u^2| - 2| \, dA = 2 \int_0^1 \int_0^{4-4u} \sqrt{v} + u^2 \, dv \, du$$

7. Notice that the surface is the boundary of the following solid E: $x^2 + y^2 \leq x \leq 4$ (the first surface is the curved outside of E, and the second surface is the flat top of E). Thus, by the divergence theorem,

$$\begin{aligned}
& \int_S \mathbf{F} \cdot d\mathbf{S} \\
= & \int_E \operatorname{div}(\mathbf{F}) \, dV \\
= & \int_D \int_{x^2+y^2}^4 (2 + 0 + 3) \, dz \, dA \\
= & 5 \int_D \int_{x^2+y^2}^4 1 \, dz \, dA \\
= & 5 \int_D 4 - (x^2 + y^2) \, dA \\
= & 5 \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta \text{ (switching to polar)} \\
= & 5 \int_0^{2\pi} \left(2r^2 - \frac{r^3}{3} \right)_0^2 d\theta \\
= & 5(2\pi)(8 - 4) \\
= & 40\pi
\end{aligned}$$

8. The complementary equation has auxiliary equation $x^2 - 1 = 0$, so has roots $x = \pm 1$. Thus the complementary equation has solution

$$y_c = c_1 e^x + c_2 e^{-x}$$

Using the method of undetermined coefficients, we try a solution of the form $y_p = e^x(a\cos x + b\sin x)$. This has derivative

$$y_p' = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x) = e^x((a+b)\cos x + (b-a)\sin x)$$

and second derivative

$$y_p'' = e^x((a+b)\cos x + (b-a)\sin x) + e^x((-a-b)\sin x + (b-a)\cos x) = e^x(2b\cos x - 2a\sin x)$$

Putting this into the given DE gives

$$e^x((2b-a)\cos + (-2a-b)\sin x) = e^x\cos x$$

Comparing coefficients, we get

$$2b - a = 1 \text{ and } -2a - b = 0$$

Solving this for a and b gives $a = \frac{-1}{5}, b = \frac{2}{5}$. Thus the general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-x} e^x \left(-\frac{1}{5}\cos x + \frac{2}{5}\sin x\right)$$

9. Since we are calculating the surface integral of the curl of a vector field, we can use Stokes' theorem. The boundary curve C of the surface S has parametrization $r(\theta) = (2\cos\theta, 2, 2\sin\theta), 0 \le \theta \le 2\pi$. The surface is oriented inwards, so

$$\begin{aligned}
& \int_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \\
= & -\int_C \mathbf{F} \cdot d\mathbf{r} \\
= & \int_C P\,dx + Q\,dy + R\,dz \\
= & -\int_0^{2\pi} (16\cos^4\theta)(-2\sin\theta) + 0 + (12 + 8\sin^3\theta)(2\cos\theta)\,d\theta \\
= & -\int_0^{2\pi} -32\sin\theta\cos^4\theta + 24\cos\theta + 16\cos\theta\sin^3\theta\,d\theta \\
= & -\left(\frac{32}{5}\cos^5\theta + 24\sin\theta + 4\sin^4\theta\right)_0^{2\pi} \\
= & \ 0
\end{aligned}$$

10. **(Bonus)** Given a continuous function f, define a vector field $\mathbf{F}$ by

$$\mathbf{F}(x, y, z) = \int_0^x f(t, y, z)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Then

$$\operatorname{div}(\mathbf{F}) = \frac{d}{dx} \int_0^x f(t, y, z) + 0 + 0 = f(x, y, z)$$

by the fundamental theorem of calculus. So $\operatorname{div}(\mathbf{F}) = f$, as required.